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AN INTEGRAL TRANSFORM TOGETHER WITH TAYLOR SERIES AND DECOMPOSITION METHOD FOR THE SOLUTION OF NONLINEAR BOUNDARY VALUE PROBLEMS OF HIGHER ORDER

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Abstract: This work aims to determine the approximate solutions of nonlinear boundary value problems of higher order obtained through the Aboodh Transform Series Decomposition Method (ATSDM), a method designed to find the integral and the inverse transform of the problems, expand the exponential function, and simultaneously decompose the nonlinear terms. The results obtained demonstrate that ATSDM is an excellent and trusted approximate method that can be employed to obtain accurate results for any problem similar to the one presented in this work.

Keywords: *Aboodh transform, adomian decomposition, taylor series, twelfth, and thirteenth order.*

1. Introduction

Higher order boundary value problems have been a major concern due to their mathematical significance or prominence, as well as their great applicability in hydrodynamics and hydromagnetics (Agarwal, 1986; Wazwaz, 2000, 2000a; Chandrasekhar, 1961; Mahdy et al., 2020; Abdel-Halim Hassan et al., 2009; Othman et al., 2010).

The analytical solution of the afore-mentioned problems, especially the nonlinear ones, have been a problem that is challenging to solve. Consequently, researchers have devised an alternative approach for obtaining an approximate solution in the literature (Oderinu, 2014; Opanuga et al., 2015, 2017; Akinola et al., 2017; Noor & Mohyud-Din, 2008; Hymavathi & Kumar, 2014; Mohyud-din, 2009; Siddiqi et al., 2009; El-Gamel, 2015; Owolabi et al., 2019; Adeyefa & Kuboye, 2020; Farooq et al., 2020; Gbadamosi et al., 2010; Amer et al., 2018; Gepreel et al., 2020; Mahdy, 2019; Mahdy et al., 2021; Mandy and Youssef, 2021).

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This research work derived its motivation from the work of Akinola et al. (2016), and extensively explained how the higher order problems considered are being transformed by the Aboodh Transformation and its inverse (Aboodh 2013, 2014; Abdelbagy & Mohand, 2016; Mahgoub & Sedeeg, 2017), how the Series Method is being employed to handle the exponential functions (Yalçinbaş, 2002), and how nonlinear terms are being decomposed by the means of Adomian Polynomials (Adomian, 1988; Abbaoui & Cherruault, 1994). The results obtained by the combination of the three mentioned methods have demonstrated the strength of the Aboodh Transform Series Decomposition Method (ATSDM) in terms of efficacy, accuracy and reliability over all other related methods available. This method is an alternative powerful mathematical tool that can be used in obtaining the solution of nonlinear differential equations of any order.

This paper is organized into five sections. Section One presents the introduction of the newly developed method, its formation, and its strength, and then reviews some of the related literature. Section Two presents the step by step approach of the said method. Section Three presents the application and implementation of the method on four different types of higher order nonlinear boundary value problems using Maple 18 software. Sections Four and Five present the conclusion and references of the literature cited, respectively.

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2. Methodology

This section examines the general nonhomogeneous nonlinear differential equation.

$$Lu(x) + Ru(x) + Nu(x) = g(x)\,. \quad (1)$$

where $Lu(x)$, $Ru(x)$, $Nu(x)$ and $g(x)$ have their own usual meaning.

Eq. (2) was obtained by applying the Aboodh Transform on Eq. (1) (Aboodh, 2013, 2014; Abdelbagy & Mohand, 2016; Mahgoub & Sedeeg, 2017):

$$A\{Lu(x)\} = A\{g(x)\} - A\{Ru(x) + Nu(x)\}\,. \quad (2)$$

The Aboodh Transformation of the derivative Eq. (2) gives:

$$A\{u(x)\} = \sum_{k=0}^{n-1} \frac{1}{v^{2-n+k}} \frac{d^n f(0)}{dx^n} + \frac{1}{v^n} A\{g(x)\} - \frac{1}{v^n} A\{Ru(x) + Nu(x)\}. \quad (3)$$

The Aboodh inverse transform of Eq. (3) now becomes:

$$u(x) = A^{-1}\left[\sum_{k=0}^{n-1} \frac{1}{v^{2-n+k}} \frac{d^n f(0)}{dx^n}\right] + A^{-1}\left[\frac{1}{v^n} A\{g(x)\}\right] - A^{-1}\left[\frac{1}{v^n} A\{Ru(x) + Nu(x)\}\right]. \quad (4)$$

Let $u(x) = \sum_{n=0}^{\infty} U_n(x)$ be an infinite series. The decomposition of the nonlinear term is now:

$$Nu(x) = \sum_{n=0}^{\infty} A_n \quad (5)$$

where A_n can be calculated as:

$$A_n = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} \left[N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}, \qquad n = 0, 1, 2 \ldots. \quad (6)$$

Substituting Eq. (6) into Eq. (4) gives:

$$\sum_{n=0}^{\infty} u_n(x) = f(x) - A^{-1}\left[\frac{1}{v^n} A\{R \sum_{n=0}^{\infty} u_n(x) + \sum_{n=0}^{\infty} A_n\}\right]. \quad (7)$$

where,

$$f(x) = A^{-1}\left[\sum_{k=0}^{n-1} \frac{1}{v^{2-n+k}} \frac{d^n f(0)}{dx^n}\right] + A^{-1}\left[\frac{1}{v^n} A\{g(x)\}\right]. \quad (8)$$

Suppose $u_0(x) = f(x)$. Then, the remaining terms $u_1(x)$, $u_2(x)$, ... was obtained as:

$$u_{n+1} = A^{-1}\left[\frac{1}{v^n} A\{R \sum_{n=0}^{\infty} u_n(x) + \sum_{n=0}^{\infty} A_n\}\right], \quad n \geq 0. \quad (9)$$

The iteration was then obtained from Eq. (9), and the solution to Eq. (1) is now;

$$u(x) = u_0 + u_1 + u_2 + u_3 + \cdots \quad (10)$$

3. Application

Illustration I: Noor & Mohyud-Din (2008) and Othman et al. (2010):

$$y^{xii} - y''' = 2e^x y^2, \qquad 0 \leq x \leq 1. \quad (11)$$

subject to the initial-boundary conditions:

$$y(0) = y''(0) = y^{iv}(0) = y^{vi}(0) = y^{vii}(0) = y^{x}(0) = 1.$$

$$y(1) = y''(1) = y^{iv}(1) = y^{vi}(1) = y^{vii}(1) = y^{x}(1) = 1.$$

The analytical solution is: $y = e^{-x}$.

Then the exponential function e^x was expanded by Taylor series to have;

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots. \quad (12)$$

Substituting the Taylor expansion Eq. (12) into Eq. (10) and following the process in Section 2, we have:

$$y = 1 - 0.9999999999999999x + \frac{1}{2}x^2 - 0.1666666666666667x^3 + \frac{1}{24}x^4 - 0.008333333333333342x^5 + \cdots.$$

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Table 1. Comparative Analysis of the Absolute Errors for Illustration I

x	Exact	VIM Noor & Mohyud-Din (2008)	HMP Othman et al., (2010)	ATSDM
0	0	0	0	0
0.2	0.818730	3.07×10^{-7}	1.172×10^{-7}	4.19×10^{-17}
0.4	0.670320	4.97×10^{-7}	8.08×10^{-7}	2.34×10^{-18}
0.6	0.548811	4.97×10^{-7}	1.19×10^{-7}	1.23×10^{-16}
0.8	0.443289	3.07×10^{-7}	5.0×10^{-10}	7.50×10^{-17}
1.0	0.367879	2.00×10^{-10}	4.1×10^{-9}	1.07×10^{-17}

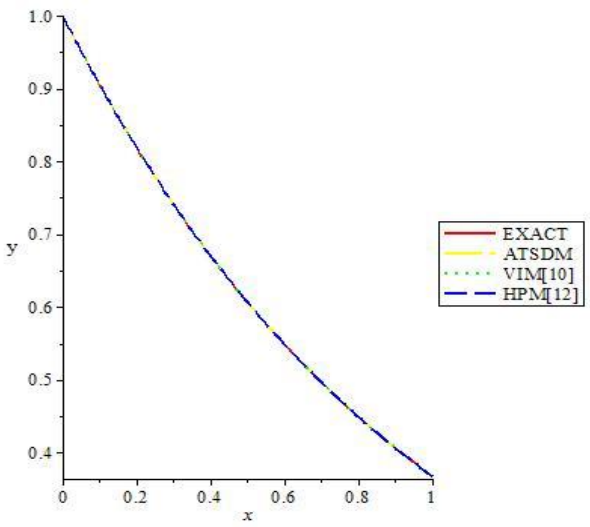

Figure 1. The Comparison between Exact, VIM, HPM, and ATSDM Solution of Illustration I

Illustration II: Considering Oderinu (2014) and Noor & Mohyud-Din (2008):

$$y^{xii} = \frac{1}{2}e^{-x}y^2, \qquad 0 \le x \le 1. \tag{13}$$

subject to

$$y(0) = y''(0) = y^{iv}(0) = y^{vi}(0) = y^{vii}(0) = y^{x}(0) = 2.$$

$$y(1) = y''(1) = y^{iv}(1) = y^{vi}(1) = y^{vii}(1) = y^{x}(1) = 2e.$$

The analytical solution was given as $y = 2e^x$.

Following the same method itemized in Illustration 1 gives:

$$y = 2 + 2.00000000000000x + x^2 + 0.3333333333333345x^3 + \frac{1}{24}x^4 + 0.01666666666666600x^5 + \cdots.$$

Table 2. Comparative Analysis of the Absolute Errors for Illustration II

x	Exact	WRM Oderinu (2014)	VIM Noor & Mohyud-Din (2008)	ATSDM
0	2.0	0	0	0
0.1	2.210341	0	2.07×10^{-4}	7.49×10^{-16}
0.2	2.442805	0	3.94×10^{-4}	3.23×10^{-16}
0.3	2.699717	1×10^{-14}	5.40×10^{-4}	2.38×10^{-16}
0.4	2.983649	0	6.32×10^{-4}	7.05×10^{-16}
0.5	3.297442	1×10^{-14}	6.61×10^{-4}	4.17×10^{-16}
0.6	3.644237	2×10^{-14}	6.26×10^{-4}	1.52×10^{-16}
0.7	4.027505	1×10^{-14}	5.31×10^{-4}	6.93×10^{-16}
0.8	4.451081	1×10^{-14}	3.84×10^{-4}	3.50×10^{-16}
0.9	4.919206	0	2.02×10^{-4}	1.01×10^{-16}
1.0	5.436564	0	2.02×10^{-4}	1.10×10^{-15}

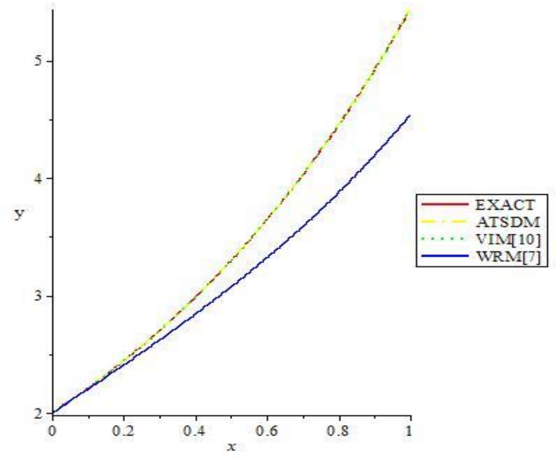

Figure 2: Comparison between Exact, VIM, WRM, and ATSDM Solution of Illustration II

Illustration III: Considering Siddiqi et al., (2009):

$$y^{xii} = (12 + x + x^2e^x)e^x - y^2, \qquad 0 \le x \le 1. \tag{14}$$

subject to the initial-boundary conditions

$$y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 2, \quad y'''(0) = 3, \quad y^{iv}(0) = 4, \quad y^{v}(0) = 5.$$

$$y(1) = e, \quad y'(1) = 2e, \quad y''(1) = 3e, \quad y'''(1) = 4e, y^{iv}(1) = 5e, \quad y^{v}(1) = 6e.$$

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Therefore, the analytical solution is given as $y = xe^x$.

Following the same method itemized in Illustration I gives:

$$y = \left(\frac{1}{25850167388849766400000}\right)\begin{pmatrix}647647525324800x^{12} + 53970627110400x^{13} \\ +4744670515200x^{14} + \cdots.\end{pmatrix}$$

Table 3. Comparative Analysis of the Absolute Errors for Illustration III

X	Exact	VIM Siddiqi et al., (2009)	ATSDM
0	0	0	0
0.1	0.1105	5.14×10^{-16}	2.39×10^{-25}
0.2	0.2443	5.27×10^{-14}	4.08×10^{-21}
0.3	0.4050	7.63×10^{-13}	1.01×10^{-16}
0.4	0.5967	4.94×10^{-12}	2.28×10^{-16}
0.5	0.8244	2.08×10^{-11}	2.81×10^{-16}
0.6	1.0933	6.69×10^{-11}	9.76×10^{-16}
0.7	1.4096	1.81×10^{-10}	3.25×10^{-17}
0.8	1.7804	4.26×10^{-10}	8.28×10^{-17}
0.9	2.2136	9.12×10^{-10}	2.67×10^{-16}
1.0	2.7183	1.81×10^{-9}	2.33×10^{-16}

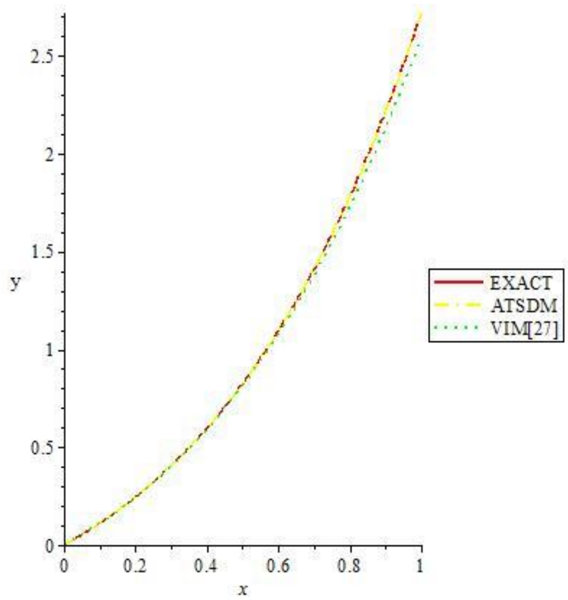

Figure 3. Comparison between Exact, VIM, and ATSDM Solution of Illustration III.

Illustration IV: Considering the work of Opanuga et al., (2017):

$$y^{xiii} = e^{-x}y^2, \qquad 0 \le x \le 1. \tag{15}$$

subject to

$$y(0) = y'(0) = y''(0) = y'''(0) = y^{iv}(0) = y^{v}(0) = y^{vi}(0) = 1.$$

$$y(1) = y'(1) = y''(1) = y'''(1) = y^{iv}(1) = y^{v}(1) = y^{vi}(1) = e.$$

The analytical solution was given as $y = xe^x$.

Following the same method itemized in Illustration I gives:

$$y = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \cdots.$$

Table 4. Comparative Analysis of the Absolute Errors for Illustration IV

x	Exact	MADM Opanuga et al., (2017)	ATSDM
0	1	0	0
0.1	1.105170	5.107×10^{-15}	3.75×10^{-16}
0.2	1.221402	3.785×10^{-13}	1.66×10^{-16}
0.3	1.349858	3.986×10^{-12}	1.06×10^{-16}
0.4	1.491824	1.772×10^{-11}	3.29×10^{-16}
0.5	1.648721	4.829×10^{-11}	1.61×10^{-16}
0.6	1.822118	9.536×10^{-11}	1.35×10^{-18}
0.7	2.013752	1.506×10^{-10}	4.42×10^{-16}
0.8	2.225540	2.042×10^{-10}	3.72×10^{-16}
0.9	2.459603	2.517×10^{-10}	3.26×10^{-16}
1.0	2.718281	2.936×10^{-10}	2.63×10^{-16}

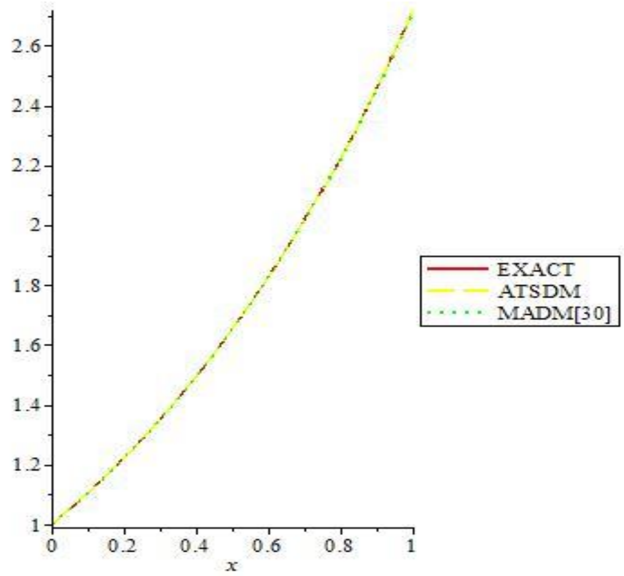

Figure 4. Comparison between Exact, MADM, and ATSDM Solution of Illustration IV.

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4. Conclusion

The comparison of the approximate solutions through ATSDM with the analytical one and other approximate methods found in the literature was shown both in tables and figures above. It can be clearly seen from the tables that the magnitude of the errors of the new approximate method presented in this work was relatively insignificant or minute when placed side by side with the one already obtained in the reviewed literature. Similarly, the figures demonstrate the rapid convergence of the method to the exact.

Therefore, the results obtained indicate that the method studied in this research is comparatively better in terms of efficiency, accuracy, simplicity, and computational cost. Hence, the ATSDM or its modification is thus recommended to researchers interested in obtaining n exact or near exact approximate solution to any nonlinear differential equations, or of the form considered in this work, irrespective of the order.

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